Dbm UNIT-I PART B (LINEAR PROGRAMMING)

(1) ~~Old hens can be bought at Rs 2 each and~~ Old hens can be bought at Rs 2 each and young ones at Rs 5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paise. A hen costs Rs 1 per week to feed. A person has ~~to~~ only Rs 80 to spend for hens, How many of each kind should he ~~try~~ buy to give a profit of more than Rs 6 per week, assuming that he cannot house more that 20 hens. Formulate this as a LPP?

<u>Soln</u>:- The person decides to buy x_1 old hens and x_2 young hens to maximize his profit.

Since he has only Rs 80 to spend for hens and old hen costs Rs 2 and young hen costs Rs 5 each.

$$2x_1 + 5x_2 \le 80.$$

Also since he cannot house more than 20 hens

$$x_1 + x_2 \le 20.$$

The total sale of eggs will be $= Rs\ 0.3\,(3x_1 + 5x_2)$

Expenditure on feeding will be $= Rs\ 1\,(x_1 + x_2)$

$\therefore$ The net profit is $= Rs\left[0.3\,(3x_1 + 5x_2) - 1(x_1 + x_2)\right]$

$$= Rs\,(0.5x_2 - 0.1x_1),$$

$$\therefore\ 0.5x_2 - 0.1x_1 \ge 6.$$

$\therefore$ The constraints are $2x_1 + 5x_2 \le 80.$

$$x_1 + x_2 \le 20.$$

$$0.5x_2 - 0.1x_2 \ge 6$$

and $x_1, x_2 \ge 0$

$\therefore$ The complete formulation of the LPP is

$$\text{Max } Z = 0.5x_2 - 0.1x_2$$

Sub to the cons/. $2x_1 + 5x_2 \le 80$, $0.5x_2 - 0.1x_2 \ge 6$

$x_1 + x_2 \le 20$ & $x_1, x_2 \ge 0$

(2) Graphical Method

Pbm: Solve the foll.. LPP by the graphical method,

Sub to $\quad -2x_1 + x_2 \leq 1$

$x_1 \leq 2$

$x_1 + x_2 \leq 3 \quad$ and $x_1, x_2 \geq 0$.

Soln: First consider the inequality constraints as equalities,

$-2x_1 + x_2 = 1$ ——— (1)

$x_1 = 2$. ——— (2) $\qquad x_1 = 0,\ x_2 = 0$

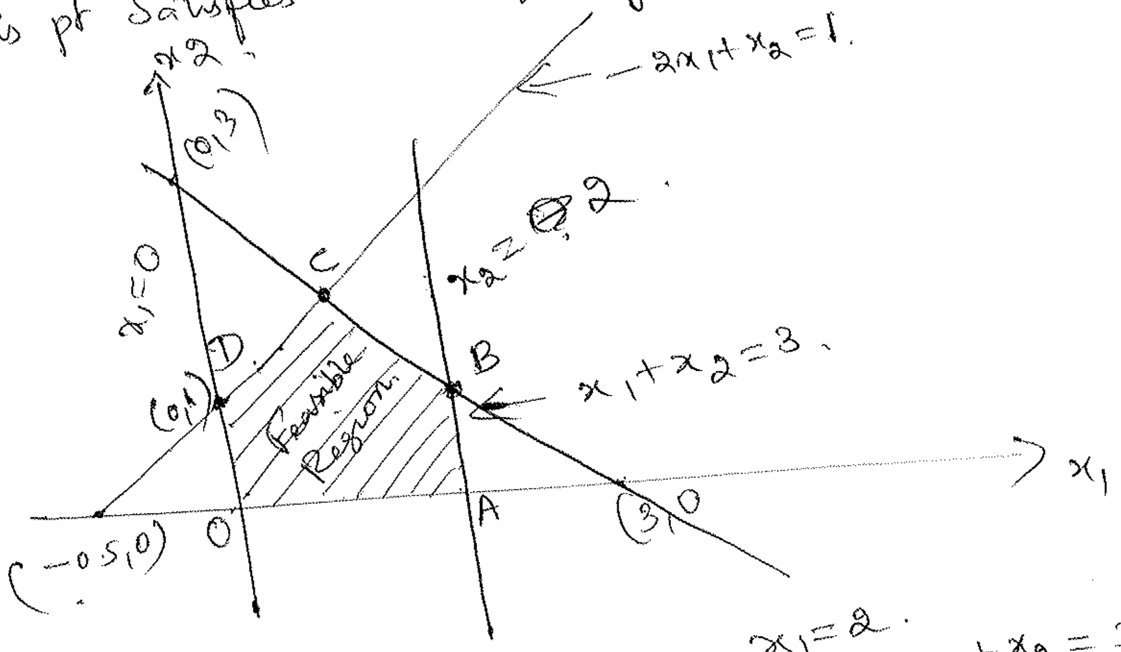
$x_1 + x_2 = 3$ ——— (3)

put $x_1 = 0$ in (1) we get $\boxed{x_2 = 1}$ $\quad \therefore$ pt is $(0,1)$

$x_2 = 0 \implies -2x_1 = 1$

$x_1 = -0.5$. pt is $(-0.5, 0)$

This pt Satisfies the inequality $2x_1 + x_2 \leq 1$.



To find the pts of intersection. $\quad x_1 = 2$, ~~$x_2 = 0$~~, $x_1 + x_2 = 3$

$\implies x_1 = 2,\ x_2 = 1$

The vertex B $(2,1)$

||ly C is the pt of intersection of $-2x_1 + x_2 = 1$, $x_1 + x_2 = 3$.

Solving these we have. $C = (2/3, 7/3)$

D $(0,1)$

The values of Z at these vertices are gn by

Vertex.	Value of Z.
O (0,0)	0
A (2,0)	6
B (2,1)	8
C (2/3, 7/3)	20/3
D (0,1)	2

Since the pbm is of maximization type, The optimum Soln to the LPP is Max $Z = 8$, $x_1 = 2$, $x_2 = 1$.

(3) Pbm 3 A Company manufactures 2 types of printed circuits. The requirements of transistors, resistors and capacitors for each type of printed circuits along with other data are given below.

	circuit A	circuit B	Stock available
Transistor	15	~~15~~ 10	180
Resistor	10	20	~~180~~ 200
Capacitor	15	20	~~220~~ 210
Profit	Rs 5	Rs 8.	

How many circuits of each type should be Company produce from the stock to earn maximum profit.

Soln: Let x_1 be the number of type A circuits and x_2 be the number of type B circuits to be produced.

The number of type B circuits

To produce these units of type A and type B circuits the company requires

Transistors $= 15x_1 + 10x_2$

Resistor $= 10x_1 + 20x_2$

Capacitors $= 15x_1 + 20x_2$.

Since the availability of these transistors, resistors and capacitors are 180, 200 and 210, The constraints are

$$15x_1 + 10x_2 \le 180$$
$$10x_1 + 20x_2 \le 200$$
$$15x_1 + 20x_2 \le 210$$
$$x_1 \ge 0,\ x_2 \ge 0$$

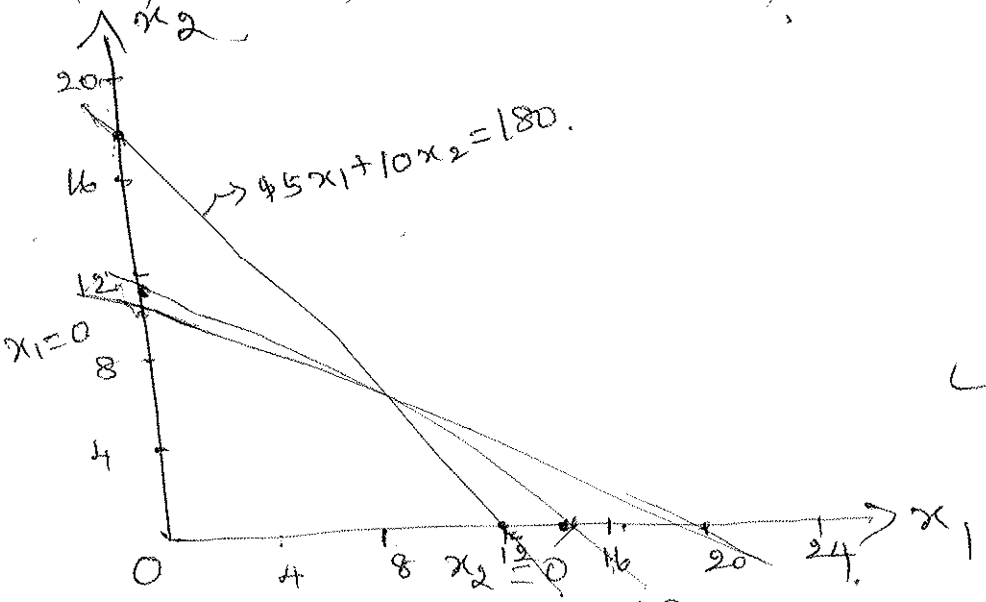
Since the profit from type A is Rs 5 and from type B is Rs 8 per units the total profit is $5x_1 + 8x_2$.

∴ The complete formulation of the LPP is

Max. $z = 5x_1 + 8x_2$

Subject to
$$15x_1 + 10x_2 \le 180 \quad \text{— (1)}$$
$$10x_1 + 20x_2 \le 200 \quad \text{— (2)}$$
$$15x_1 + 20x_2 \le 210 \quad \text{— (3)}$$
$$x_1, x_2 \ge 0.$$

By using graphical Method, the soln space is gn below with shaded area OABCD with vertices O(0,0), A(12,0), B(10,3), C(2,9), D(0,10)



14
~~70~~
~~210~~
15

put
$x_1 = 0 \quad x_2 = 18 \qquad (0, 18)$
$x_1 = 12, \quad x_2 = 0 \qquad (12, 0)$ } ⇒ $15x_1 + 10x_2 = 180$

$x_1 = 0, \quad x_2 = 10 \qquad (0, 10)$
$x_2 = 0 \quad x_1 = 20 \qquad (20, 0)$ } $10x_1 + 20x_2 = 200$.

$x_1 = 0 \quad x_2 = 10.5 \qquad (0, 10.5)$
$x_2 = 0, \quad x_1 = 14 \qquad (14, 0)$ } $15x_1 + 20x_2 = 210$.

The values of z of these vertices are gn by

Vertex	Value of z
O (0,0)	0
A (12,0)	60
B (10,3)	74
C (2,9)	82
D (0,10)	80

Since the pbm is of maximization type, the optimum soln is

Max $z = 82$, $x_1 = 2$, $x_2 = 9$.

HW Pbm 4. Apply graphical method to solve the LPP. Maximize $z = x_1 - 2x_2$ sub. to $-x_1 + x_2 \le 1$, $6x_1 + 4x_2 \ge 24$, $0 \le x_1 \le 5$ and $2 \le x_2 \le 4$.

Simplex Method

(5) Max $z = 4x_1 + 10x_2$.

Sub to
$$2x_1 + x_2 \le 50$$
$$2x_1 + 5x_2 \le 100$$
$$2x_1 + 3x_2 \le 90 \text{ and } x_1, x_2 \ge 0$$

Soln:- By introducing the slack variables s_1, s_2 and s_3 the pbm in std form becomes

$$\text{Max } z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3.$$

Sub. to.
$$2x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 50.$$
$$2x_1 + 5x_2 + 0s_1 + s_2 + 0s_3 = 100.$$
$$2x_1 + 3x_2 + 0s_1 + 0s_2 + s_3 = 90$$
and $x_1, x_2, s_1, s_2, s_3 \ge 0$.

Since there are 3 eqns with 5 variables, the initial basic feasible soln is obtained by equating $(5-3) = 2$ variables to zero.

∴ The initial basic feasible soln is $s_1 = 50$, $s_2 = 100$, $s_3 = 90$

The initial simplex table is

c_j (4 10 0 0 0)

C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	$\theta = \min \frac{X_{Bi}}{a_{ir}}$
0	s_1	50	2	1	1	0	0	50
0	s_2	100	2	(5)	0	1	0	20*
0	s_3	90	2	3	0	0	1	30
$z_j - c_j$		0	−4	−10	0	0	0	

The net evaluations are calculated as $z_j - c_j = C_B a_j - c_j$

$z_1 - c_1 = C_B a_1 - c_1 = \cancel{(0, 0, 0)} (0\ 0\ 0) \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} - 4 = -4$.

$z_2 - c_2 = C_B a_2 - c_2 = (0, 0\ 0)(1\ 5\ 3)^T - 10 = -10$.

$z_3 - c_3 = C_B a_3 - c_3 = (0\ 0\ 0)(1\ 0\ 0)^T - 10 = 0$

$z_4 - c_4 = C_B a_4 - c_4 = (0\ 0\ 0)\ (0\ 1\ 0)^T - 0 = 0$

$z_5 - c_5 = C_B a_5 - c_5 = (0\ 0\ 0)(0\ 0\ 1)^T - 0 = 0$

Since there are some $(z_j - c_j) < 0$ the current basic feasible soln is not optimal

To find the entering variable

Since $z_2 - c_2 = -10$ is the most negative, the corresponding non basic variable x_2 enters the basis. The column corresponding to this x_2 is called the key column (or) pivot column

To find the leaving variable

Find the ratio $\theta = \min \left\{ \frac{X_{Bi}}{a_{ir}},\ a_{ir} > 0 \right\}$

$= \min \left\{ \frac{X_{Bi}}{a_{i2}},\ a_{i2} > 0 \right\}$

$\theta = \min \left\{ \frac{50}{1}, \frac{100}{5}, 90/3 \right\}$

$= \min \{50, 20, 30\} = 20$ which corresponds to s_2

$\therefore$ The leaving vari/. is the basic vari s_2 which corresponds to the min ratio $\theta = 20$. The leaving variable row is called the pivot row or keyrow or pivot eqn and 5 is the pivot elt. Now

New pivot eqn = old pivot eqn $\div$ pivot elt

$$= (100 \;\; 2 \;\; 5 \;\; 0 \;\; 1 \;\; 0) \div 5$$

$$= (20 \;\; 2/5 \;\; 1 \;\; 0 \;\; 1/5 \;\; 0)$$

New s_1 eqn = old s_1 eqn $-$ (corresponding column coefficient) $\times$ (New pivot eqn)

=	50	2	1	1	0	0
(−)	20	2/5	1	0	1/5	0
	30.	8/5	0	1	−1/5	0

New s_3 eqn =	90	2	3	0	0	1
(−)	60	6/5	3	0	3/5	0
	30	4/5	0	0	−3/5	1

New $(z_j - c_j)$ eqn =	0	−4	−10	0	0	0
(−) =	−200	−20/5	−10	0	−10/5	0
	200	0	0	0	2	0

$\therefore$ The improved basic feasible soln is gn in the foll simplex table.

First iteration

	c_j		4	10	0	0	0
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
0	s_1	30	$8/5$	0	1	$-1/5$	0
10	x_2	20	$2/5$	1	0	$1/5$	0
0	s_3	30	$4/5$	0	0	$-3/5$	1
$Z_j - c_j$		200	0	0	0	2	0

Since all $Z_j - c_j \geq 0$ the current basic feasible soln is optimal

$\therefore$ The optimal soln is Max $z = 200$, $x_1 = 0$, $x_2 = 20$

HW. Ex 6. Find the non negative values of x_1, x_2 and x_2 which maximize $Z = 3x_1 + 2x_2 + 5x_3$

Sub to
$$x_1 + 4x_2 \leq 420$$
$$3x_1 + 2x_3 \leq 460$$
$$x_1 + 2x_2 + x_3 \leq 430$$

7. A gear manufacturing company received an order for three specific types of gears for regular supply. The management is considering to devote the available excess capacity to one or more of the three types, say A, B and C. The available capacity on the machine which might limit output and the number of machine hours required for each unit of the respective gear is also given below.

Machine Type	Available machine hours/week	Productivity in machine hours/unit		
		Gear A	Gear B	Gear C
Gear Hobbing m/c	250	8	2	3
Gear Shaping m/c	150	4	3	0
Gear Grinding m/c	50	2	—	1

The unit profit would be Rs 20, Rs 6 and Rs 8 respectively for the gears A, B and C. Find how much of each gear the company should produce in order to maximize profit?

Soln: Let x_1, x_2 and x_3 be the number of units of gears A, B and C produced respectively to maximize the profit. The mathematical formulation of the LPP is given by

$$\text{Maximize } Z = 20x_1 + 6x_2 + 8x_3.$$

Subject to $8x_1 + 2x_2 + 3x_3 \leq 250.$

$4x_1 + 3x_2 \leq 150.$

$2x_1 + x_3 \leq 50$ and $x_1, x_2, x_3 \geq 0.$

By introducing non negative slack variables s_1, s_2 and s_3 the standard form of the LPP becomes

$$\text{Maximize } z^* = 20x_1 + 6x_2 + 8x_3 + 0s_1 + 0s_2 + 0s_3$$

Subject to the constraints

$$8x_1 + 2x_2 + 3x_3 + s_1 + 0s_2 + 0s_3 = 250$$

$$4x_1 + 3x_2 + 0s_1 + s_2 + 0s_3 = 150$$

$$2x_1 + 0x_2 + x_3 + 0s_1 + 0s_2 + s_3 = 50$$

and $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$.

∴ The initial basic feasible solution is given by

$s_1 = 250$, $s_2 = 150$, $s_3 = 50$. ($x_1 = x_2 = x_3 = 0$, non basic)

Initial iteration:

		c_j	20	6	8	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
0	s_1	250	8	2	3	1	0	0	250/8
0	s_2	150	4	3	0	0	1	0	150/4
0	s_3	50	(2)	0	1	0	0	1	50/2 *
$z_j - c_j$		0	−20	−6	−8	0	0	0	

First iteration: Introduce x_1 and drop s_3

		c_j	20	6	8	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
0	s_1	50	0	−2	−1	1	0	−4	50/2
0	s_2	50	0	(3)	−2	0	1	−2	50/3 *
20	x_1	25	1	0	1/2	0	0	1/2	—
$z_j - c_j$		500	0	−6	2	0	0	10	

Second iteration : Introduce x_2 and drop s_2.

	c_j		20	6	8	0	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
0	s_1	$50/3$	0	0	$1/3$	1	$-2/3$	$-8/3$	50
6	x_2	$50/3$	0	1	$-2/3$	0	$1/3$	$-2/3$	–
20	x_1	25	1	0	($1/2$)	0	0	$1/2$	50^*
$Z_j - C_j$		600	0	0	-2	0	2	6	

Third iteration : Introduce x_3 and drop x_1

	c_j		20	6	8	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
0	s_1	0	$-2/3$	0	0	1	$-2/3$	-3
6	x_2	50	$4/3$	1	0	0	$1/3$	0
8	x_3	50	2	0	1	0	0	1
$Z_j - C_j$		700	4	0	0	0	2	8

Since all $Z_j - C_j \geq 0$ the current basic feasible solution is optimal.

$\therefore$ The optimal soln is Max $Z = 700$, $x_1 = 0$, $x_2 = 50$, $x_3 = 50$.

$\therefore$ The company should produce 50 units of gear B 50 units of gear of C and none of gear A in order to have a maximum profit Rs. 700

8\. Solve by Big-M method

Minimize $Z = 4x_1 + 3x_2$

Subject to $2x_1 + x_2 \geq 10$,

$-3x_1 + 2x_2 \leq 6$, $x_1 + x_2 \geq 6$ and $x_1, x_2 \geq 0$

Soln:

Given Min $z = 4x_1 + 3x_2$

Subject to $2x_1 + x_2 \geq 10$, $-3x_1 + 2x_2 \leq 6$, $x_1 + x_2 \geq 0$

$x_1, x_2 \geq 0$

Max $z^* = -4x_1 - 3x_2$

Subject to $2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$, $x_1, x_2 \geq 0$

By introducing the non negative slack, surplus and artificial variables the standard form of the LPP becomes

$$\text{Max } z^* = -4x_1 - 3x_2 + 0s_1 + 0s_2 + 0s_3 - MR_1 - MR_2$$

Subject to

$$2x_1 + x_2 - s_1 + 0s_2 + 0s_3 + R_1 = 10,$$

$$-3x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 6,$$

$$x_1 + x_2 + 0s_1 + 0s_2 - s_3 + R_2 = 6,$$

and $x_1, x_2, x_3, s_1, s_2, s_3, R_1, R_2 \geq 0$

The initial basic feasible soln is given by

$R_1 = 10$, $s_2 = 6$, $R_2 = 6$ (basic) ($x_1 = x_2 = s_1 = s_3 = 0$, non-basic)

		c_j	-4	3	0	0	0	$-M$	$-M$	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_1	R_2	θ
$-M$	R_1	10	(2)	1	-1	0	0	1	0	$10/2 = 5$
0	s_2	6	-3	2	0	1	0	0	0	—
$-M$	R_2	6	1	1	0	0	-1	0	1	$6/1 = 6$
$Z_j^* - c_j$		$-16M$	$-3M+4$	$-2M+3$	M	0	M	0	0	

Since there are some $z_j^* - c_j < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_1 enters into the basis and the basic variable R_1 leaves the basis.

First iteration:

		c_j	-4	-3	0	0	0	$-M$	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_2	θ
-4	x_1	5	1	$1/2$	$-1/2$	0	0	0	10
0	s_2	21	0	$7/2$	$-3/2$	1	0	0	$42/7$
$-M$	R_2	1	0	$(1/2)$	$1/2$	0	-1	1	2
$z_j^* - c_j$		$-M-20$	0	$\frac{-M+2}{2}$	$\frac{-M+4}{2}$	0	M	0	

Since there are some $z_j^* - c_j < 0$ the current basic feasible solution is not optimal.

The non-basic variable x_2 enters and the basic variable R_2 leaves the basis.

Second iteration:

		c_j	-4	-3	0	0	0
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
-4	x_1	4	1	0	-1	0	1
0	s_2	14	0	0	-5	1	7
-3	x_2	2	0	1	1	0	-2
$z_j^* - c_j$		-22	0	0	1	0	2

Since all $z_j^* - c_j \geq 0$ the current basic feasible solution is optimal.

$\therefore$ Max $z^* = -22$, $x_1 = 4$, $x_2 = 2$

But Min z = $-\text{Max}(-z)$ = $-\text{Max } z^* = -(-22)$
$= 22.$

$\therefore$ The optimal solution is

Min $z = 22, x_1 = 4, x_2 = 2.$

9. Solve by two phase simplex method.

Maximize $X_0 = -4x_1 - 3x_2 - 9x_3$

Subject to

$$2x_1 + 4x_2 + 6x_3 - s_1 + R_1 = 15$$
$$6x_1 + x_2 + 6x_3 - s_2 + R_2 = 12$$
$$x_1, x_2, x_3, s_1, s_2, R_1, R_2 \geq 0$$

Soln: The initial basic feasible soln is given by $R_1 = 15$, $R_2 = 12$ (basic) ($x_1 = x_2 = x_3 = s_1 = s_2 = 0$) non basic.

Phase I Assigning a cost -1 to the artificial variables and costs 0 to all other variables, the objective function of the auxillary LPP becomes

$$\text{Max } z^* = -R_1 - R_2$$

Initial iteration:

		c_j	0	0	0	0	0	-1	-1	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	R_1	R_2	θ
-1	R_1	15	2	4	6	-1	0	1	0	$15/6$
-1	R_2	12	6	1	(6)	0	-1	0	1	$12/6$
$z_j^* - c_j$		-27	-8	-5	-12	1	1	0	0	

Since there are some $(z_j^* - c_j) < 0$, the current basic feasible solution is not optimal.

First iteration: Introduce x_3 and drop R_2.

C_B	Y_B	X_B	x_1 (0)	x_2 (0)	x_3 (0)	s_1 (0)	s_2 (0)	R_1 (−1)	R_2 (−1)	θ
−1	R_1	3	−4	(3)	0	−1	1	1	−1	3/3
0	x_3	2	1	1/6	1	0	−1/6	0	1/6	12
$z_j^* - c_j$		−3	4	−3	0	1	−1	0	2	

Since there are some $(z_j^* - c_j) < 0$ the current basic feasible solution is not optimal.

Second iteration: Introduce x_2 and drop R_1.

C_B	Y_B	X_B	x_1 (0)	x_2 (0)	x_3 (0)	s_1 (0)	s_2 (0)	R_1 (−1)	R_2 (−1)
0	x_2	1	−4/3	1	0	−1/3	1/3	1/3	−1/3
0	x_3	11/6	22/18	0	1	1/18	−4/18	−1/18	4/18
$z_j^* - c_j$		0	0	0	0	0	0	1	1

Since all $z_j^* - c_j \geq 0$ the current basic feasible solution is optimal. Further no artificial appears in the basis, so we proceed to phase II.

Phase II: Here we consider the actual costs associated with the original variables. The

new objective function

$$\text{Max } X_0 = -4x_1 - 3x_2 - 9x_3 + 0s_1 + 0s_2.$$

The initial basic feasible soln for this phase is the one obtained at the end of phase I.

Initial iteration :-

		C_j	-4	-3	-9	0	$0.$	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	$\theta.$
-3	x_2	1	$-4/3$	1	0	$-1/3$	$1/3$	$-$
-9	x_3	$11/6$	$\frac{22}{18}$	0	1	$1/18$	$-4/18$	$3/2$
$(X_0 - C_j)$		$-39/2.$	-3	0	0	$1/2$	$1.$	

Since there are some $(Z_j - C_j) \leq 0$ the current basic feasible soln is not optimal.

C_B	Y_B	X_B	x_1	x_2	x_3	s_1	$s_2.$
-3	x_2	3	0	1	$12/11$	$-3/11$	$1/11.$
-4	x_1	$3/2.$	1	0	$18/22,$	$1/22$	$-4/22.$
$(X_0 - C_j)$		-15	0	0	$27/11$	$7/11.$	$5/11.$

Since all $(X_0 - C_j) \geq 0$, the current basic feasible soln is optimal.

$\therefore$ The optimal soln is Max $X_0 = -15$, $x_1 = 3/2$, $x_2 = 3$ $x_3 = 0.$

10. Solve the following LPP by simplex method.

Max $z = 2x_1 + x_2$.

Subject to the constraints $x_1 - x_2 \le 10$

$2x_1 - x_2 \le 40$

and $x_1, x_2 \ge 0$.

Soln: By introducing the non-negative slack variables s_1, s_2 the standard form of the LPP becomes,

Max $z = 2x_1 + x_2 + 0s_1 + 0s_2$.

Subj to: $x_1 - x_2 + s_1 + 0s_2 = 10$

$2x_1 - x_2 + 0s_1 + s_2 = 40$

$x_1, x_2, s_1, s_2 \ge 0$.

		C_j	2	1	0	0	
Q	CV	C	x_1	x_2	s_1	s_2	θ
10	s_1	0	(1)	-1	1	0	$\frac{10}{1}$ *
40	s_2	0	2	-1	0	1	$40/2$
0	$Z_j - C_j$		-2	-1	0	0	

Since there are some $Z_j - C_j < 0$, the current basic feasible solution is not optimal. The non basic variable x_1 enters into the basis and the basic variable s_1 leaves the basis.

			2	1	0	0	
Q	CV	C	x_1	x_2	s_1	s_2	θ
10	x_1	2	1	-1	1	0	—
20	s_2	0	0	(1)	-2	1	$\frac{20}{1}$ *
20	$Z_j - C_j$		0	-3	2	0	

Since $z_2 - c_2 = -3 < 0$, the current basic feasible soln is not optimal. The non-basic variable x_2 enters into the basis and the basic variable s_2 leaves the basis

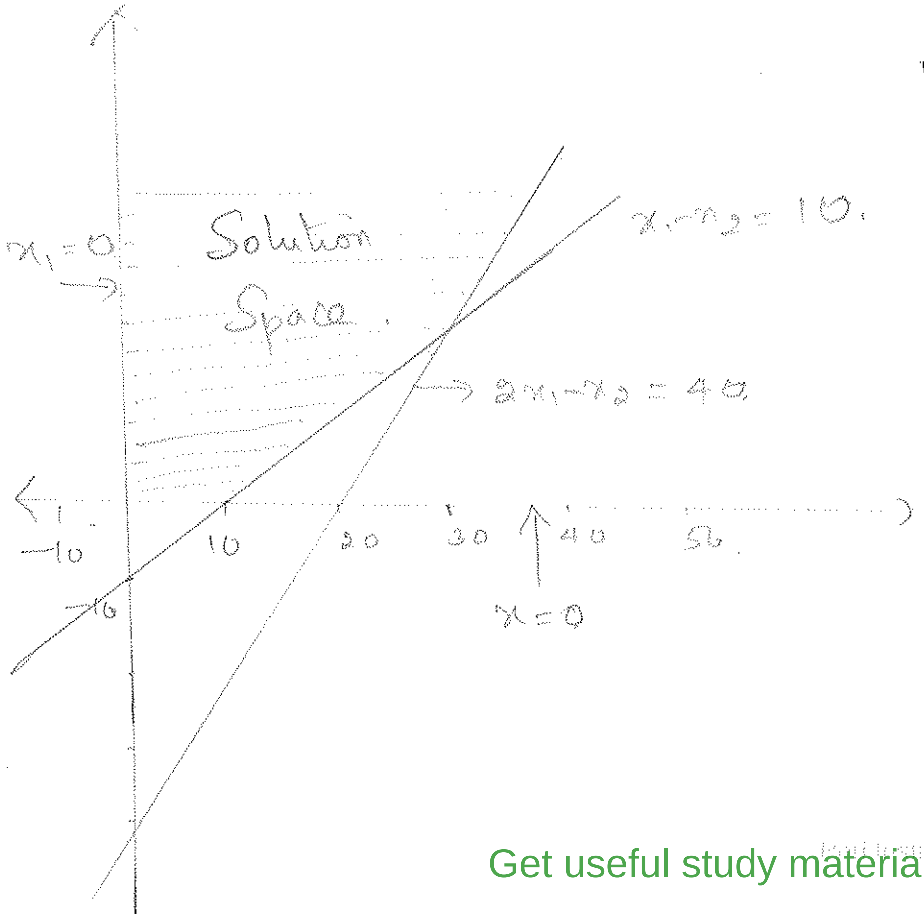
Second iteration:

		c_j	2	1	0	0
Q	CV	C.	x_1	x_2	s_1	s_2
30	x_1	2	1	0	-1	1
20	x_2	1	0	1	-2	1

Since $z_3 - c_3 = -4 < 0$, the current basic feasible soln is not optimal.

Since all $a_{ir} < 0$ it is not possible to find the positive ratio $\theta = \min\left\{ X_{Bi}/a_{ir} ; a_{ir} > 0 \right\}$. It is not possible to find the ~~pos~~ leaving variable. The solution of this problem is unbounded.

The problem is represented graphically to indicate that there is a unbounded optimal solution with unbounded solution space.



Here the solution space is unbounded and the optimal solution is also unbounded.

Solve

11 Max $z = 8x_2$.

$x_1 - x_2 \geq 0$

$2x_1 + 3x_2 \leq -6$

x_1, x_2 are unrestricted.

Soln:

Since x_1, x_2 are unrestricted, we put $x_1 = x_1' - x_1''$ and $x_2 = x_2' - x_2''$ so that $x_1', x_1'', x_2', x_2'' \geq 0$.

∴ The given LPP becomes

$$\text{Max } z = 8x_2' - 8x_2''$$

Sub to $x_1' - x_1'' - x_2' + x_2'' \geq 0$

$-2x_1' + 2x_1'' - 3x_2' + 3x_2'' \geq 6$

$x_1', x_1'', x_2', x_2'' \geq 0$

By introducing the surplus variables s_1, s_2 and artificial variables R_1 and R_2.

$$\text{Max } z = 0x_1' + 0x_1'' + 8x_2' - 8x_2'' + 0s_1 + 0s_2 - MR_1 - MR_2.$$

Sub to $x_1' - x_1'' + x_2' + x_2'' - s_1 + 0s_2 + R_1 = 0$.

$-2x_1' + 2x_1'' - 3x_2' + 3x_2'' + 0s_1 - s_2 + R_2 = 6$

$x_1', x_1'', x_2', x_2'', s_1, s_2, R_1, R_2 \geq 0$.

The initial basic feasible soln is gn by $R_1 = 0$, $R_2 = 6$.

Initial iteration:

			0	0	8	−8	0	0	−M	−M	
Q	CV	C	x_1'	x_1''	x_2'	x_2''	s_1	s_2	R_1	R_2	θ
0	R_1	−M	1	−1	−1	(1)	−1	0	1	0	0*
6	R_2	−M	−2	2	−3	3	0	−1	0	1.	6/3 = 2
−6M	$Z_j - C_j$		M	−M	+4M−8	−M+8	M	M	0	0	

First iteration : Introduce x_2'' and drop R_1.

Q	CV	C	x_1'	x_1''	x_2'	x_2''	s_1	s_2	R_2	θ
c_j			0	0	8	-8	0	0	-M	
0	x_2''	-8	1	-1	-1	1	-1	0	0	—
6	R_2	-M	-5	(5)	0	0	3	-1	1	$6/5$*
$Z_j - C_j$		-6M	5M-8	-5M+8	0	0	-3M+8	M	0	

Second iteration : Introduce x_1'' and drop R_2.

Q	CV	V	x_1'	x_1''	x_2'	x_2''	s_1	s_2
$6/5$	x_2''	-8	0	0	-1	1	$-2/5$	$-1/5$
$6/5$	x_1''	0	-1	1	0	0	$3/5$	$-1/5$
$-48/5$	$Z_j - C_j$		0	0	0	0	$16/5$	$8/5$

Since all $Z_j - C_j \geq 0$ and no artificial variable appears in the basis.

$\therefore$ The optimal soln Max $z = -48/5$, $x_1' = 0$, $x_2' = 0$

$x_1'' = 6/5$, $x_2'' = 6/5$

$x_1 = x_1' - x_1'' = -6/5$

$x_2 = x_2' - x_2'' = -6/5$.

The optimal soln is Max $z = -48/5$.

12. Max $Z = x_1 + 2x_2$

sub to

$$x_1 + x_2 \le 3$$
$$x_1 + 2x_2 \le 5$$
$$3x_1 + x_2 \le 6, \quad x_1, x_2 \geq 0$$

Soln:

$$\text{Max } z = x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3$$

sub to

$$x_1 + x_2 + s_1 = 3$$
$$x_1 + 2x_2 + s_2 = 5$$
$$3x_1 + x_2 + s_3 = 6$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Standard form - I

$$Z - x_1 - 2x_2 - 0s_1 - 0s_2 - 0s_3 = 0$$
$$x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 3$$
$$x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 5$$
$$3x_1 + x_2 + 0s_1 + 0s_2 + s_3 = 6, \quad x_1, x_2, s_1, s_2, s_3 \geq 0$$

Matrix form.

$$\begin{array}{cccccc} \beta_0^{(1)} & & & \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} \\ e_1 & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & a_5^{(1)} \end{array}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 5 \\ 6 \end{bmatrix}$$

Revised Simplex table	B_1^{-1}						
Basic Variables	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	X_B	X_K	X_B/X_K
Z	1	0	0	0	0		
s_1	0	1	0	0	3		
s_2	0	0	1	0	5		
s_3	0	0	0	1	6		

Additional table

$a_1^{(1)}$	$a_2^{(1)}$
-1	-2
1	1
1	2
3	1

Computation of Δ_j for $a_1^{(1)}$ and ~~to~~ $a_2^{(1)}$.

Δ_1 = first row of $B_1^{-1} * a_1^{(1)} = 1*(-1)+0*1+0*1+0*3 = -1$.

Δ_2 = first row of $B_1^{-1} * a_2^{(1)} = 1*(-2)+0*1+0*2+0*1 = -2$

$\Delta_2 = -2$ most negative, so $a_2^{(1)}$ (x_2) is incoming vector

Compute the column vector X_K.

$X_K = B_1^{-1} * a_2^{(1)}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Basic Variable					X_B	X_K	X_B / X_K
Z	1	0	0	0	0	-2	
S_1	0	1	0	0	3	1	3
S_2	0	0	1	0	5	$\boxed{2}$	5/2 →
S_3	0	0	0	1	6	1	6

Improved Solution.

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	X_B	X_k
R_1	0	0	0	0	-2
R_2	1	0	0	3	1
R_3	0	1	0	5	2
R_4	0	0	1	6	1

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	X_B	X_k
R_1	0	1	0	5	0
R_2	1	$-1/2$	0	$1/2$	0
R_3	0	$1/2$	0	$5/2$	1
R_4	0	$-1/2$	1	$7/2$	0

Revised simplex table for II iteration

Basic Variables		e_1	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	X_B	X_k	X_B/X_k
	Z	1	0	1	0	5		
S_1		0	1	$-1/2$	0	$1/2$		
x_2		0	0	$1/2$	0	$5/2$		
S_3		0	0	$-1/2$	1	$7/2$		

$a_1^{(1)}$	$a_4^{(1)}$
-1	0
1	0
1	1
3	0

$$\Delta_1 = 1*(-1)+0*1+1*1+0*3=0$$

$$\Delta_4 = 1*0+0*0+1*1+0*0=1$$

Δ_1 and Δ_4 are +ve. Therefore optimal soln is

$$\text{Max } Z = 5,\ x_1=0,\ x_2=5/2$$

Part B. UNIT IV

CLASSICAL OPTIMISATION THEORY

1. Investigate $f(x) = x^4 - 2x^2 - 16x + 1$ for maxima and minima, use Newton – Raphson method to determine the extreme values correct to 5 decimal places.

Soln:

$$f(x) = x^4 - 2x^2 - 16x + 1.$$

$$f'(x) = 4x^3 - 4x - 16$$

$f'(x) = 0$ gives. $4x^3 - 4x - 16 = 0$

$$x^3 - x - 4 = 0$$

using Descarte's rule of signs there is atmost one positive root and no negative root. Two other roots are complex. The positive root lies between 1 and 2.

$$f'(1) < 0 \text{ and } f'(2) > 0$$

This is a real root between 1 and 2. Take the initial approximation as $x_0 = \frac{1+2}{2} = 1.5$ and use the Newton – Raphson formula.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots$$

$$x_{k+1} = x_k - \frac{(x_k^3 - x_k - 4)}{3x_k^2 - 1} = \frac{2x_k^3 + x_k + 4}{3x_k^2 - 1}.$$

x_k	$x_{k+1} = \frac{2x_k^3 + x_k + 4}{3x_k^2 - 1}$
x_0	1.5
x_1	1.87
x_2	1.80
x_3	1.796
x_4	1.796

The only extreme value is 1.796 correct to is 3 decimal places since two consecutive iterations agree at $k=3,4$.

$$f''(x) = 12x^2 - 4$$

$$f''(1.796) = 12 \times (1.796)^2 - 4 > 0.$$

$$\therefore x = 1.796 \text{ minimizes } f(x)$$

Minimum value of $f(x) = (1.796)^4 - 2(1.796)^2 - 16(1.796) + 1$

$$= -23.783.$$

2. Investigate for maxima, minima and saddle point for the function $Z = x^4 + y^4 - x^2 - y^2 + 1$.

Soln:

$$Z = x^4 + y^4 - x^2 - y^2 + 1.$$

$$\frac{\partial Z}{\partial x} = 4x^3 - 2x.$$

$$\frac{\partial Z}{\partial y} = 4y^3 - 2y.$$

$\frac{\partial Z}{\partial x} = 0$ gives $4x^3 - 2x = 0.$

$$2x(2x^2 - 1) = 0$$

$$x = 0, \quad x = \pm\frac{1}{\sqrt{2}}.$$

$\frac{\partial z}{\partial y} = 0$ gives $4y^3 - 2y = 0$

$$2y(2y^2 - 1) = 0$$

$$y = 0, \quad y = \pm\frac{1}{\sqrt{2}}.$$

$$r = \frac{\partial^2 Z}{\partial x^2} = 12x^2 - 2.$$

$t = \frac{\partial^2 z}{\partial y^2} = 12y^2 - 2.$

$s = \frac{\partial^2 z}{\partial x \partial y} = 0$

$rt - s^2 = 4(6x^2-1)(6y^2-1)$

The stationary points are $(0,0), (0, \frac{1}{\sqrt{2}}), (0, -1/\sqrt{2})$

$(\frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -1/\sqrt{2}), (-1/\sqrt{2}, 0), (-1/\sqrt{2}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

Case (i). At $(0,0)$ $rt - s^2 = 4 > 0$

$r = -2 < 0$

$\therefore$ z is maximum at $(0,0)$ and $\max z = 1$.

Case (ii) $(0, \frac{1}{\sqrt{2}})$, $rt - s^2 = -8 < 0.$

$\therefore (0, 1/\sqrt{2})$ is a Saddle point.

Similarly $(0, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, 0), (-1/\sqrt{2}, 0)$ are all saddle points.

Case (iii) At $(\pm \frac{1}{\sqrt{2}}, \pm 1/\sqrt{2})$.

$rt - s^2 = 16 > 0$, $r = 2 > 0.$

z is minimum at $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$

Minimum Value of $z = 1.$

3. Find the shortest distance from the origin to the surface $xyz^2 = 2$.

Soln: Distance of any point (x, y, z) on the surface from

the origin is given by $\sqrt{x^2+y^2+z^2}$

Let $d = \sqrt{x^2+y^2+z^2}$

$$d^2 = x^2+y^2+z^2$$

If d^2 is minimum, d is minimum, But (x,y,z) is an point on the surface.

$\therefore$ x, y should have the same sign since $z^2 = \frac{2}{xy} > 0$

$\therefore$ The stationary points are $(1,1)$ and $(-1,-1)$

(ie) $(1,1,\sqrt{2})$, $(-1,-1,\sqrt{2})$ in space.

$$r = \frac{\partial^2 f}{\partial x^2} = 2 + \frac{4}{x^3 y} = 6 > 0 \text{ at } (1,1) \;\&\; (-1,-1)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{2}{x^2y^2} = 2 \text{ at } (1,1) \text{ and } (-1,-1)$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2 + \frac{4}{xy^3} = 6 \text{ at } (1,1) \text{ and } (-1,-1)$$

$rt - s^2 = 36 - 4 = 32 > 0$, $r > 0$.

$x = 1$, $y = 1$ minimises f.

$x = -1$, $y = -1$ also minimises f.

Min $f = \sqrt{x^2+y^2+z^2} = \sqrt{1+1+2} = \sqrt{4} = 2$.

Shortest distance $x = 2$ units.

4. Find the dimensions of a rectangular parallelepiped with largest volume whose side are parallel to the coordinate planes to be inscribed in the ellipsoid.

Soln: $G(x,y,z) = x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 = 0$

Soln:
Let $2x, 2y, 2z$ be the sides of the inscribed parallelepiped then the volume of such a parallelepiped

$$V = 8xyz \quad \text{--- (1)}$$

Consider the Lagrangean function

$$L = V - \lambda G(x, y, z)$$

(ie) $$L = 8xyz - \lambda\left(x^2/a^2 + y^2/b^2 + z^2/c^2 - 1\right)$$

Extreme points are given by

$$\frac{\partial L}{\partial x} = 8yz - \lambda\left(\frac{2x}{a^2}\right) = 0 \quad \text{--- (2)}$$

$$\frac{\partial L}{\partial y} = 8xz - \lambda\left(\frac{2y}{b^2}\right) = 0 \quad \text{--- (3)}$$

$$\frac{\partial L}{\partial z} = 8xy - \lambda\left(\frac{2z}{c^2}\right) = 0 \quad \text{--- (4)}$$

and $x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 = 0.$

(1) $\times x$ + (2) $\times y$ + (3) $\times z$ gives

$$24xyz - 2\lambda\left(x^2/a^2 + y^2/b^2 + \frac{z^2}{c^2}\right) = 0 \quad \text{--- (5)}$$

$$24xyz - 2\lambda = 0$$

$$\lambda = 12xyz \quad \text{--- (6)}$$

we have $8yz - \lambda\left(\frac{2x}{a^2}\right) = 0$

Substitute (6) we get $8yz - \frac{12xyz(2x)}{a^2} = 0$

$$8yz\left(1 - 3x^2/a^2\right) = 0$$

$1 - 3x^2/a^2 = 0$

(ie) $x = a/\sqrt{3}$

Similarly $y = b/\sqrt{3}$ and $z = c/\sqrt{3}$

$\therefore$ Volume $= \dfrac{8abc}{3\sqrt{3}}$ cubic units.

5 Maximize $Z = 8x_1 + 10x_2 - x_1^2 - x_2^2$

Subject to $3x_1 + 2x_2 \le 6$

$x_1, x_2 \geq 0$

Soln:

Let $f(x) = 8x_1 + 10x_2 - x_1^2 - x_2^2$

$h(x) = 3x_1 + 2x_2 - 6$

Kuhn-Tucker conditions for the maximisation problem

because $f(x) - \lambda h(x) = 0,\ j = 1, 2.$

$\lambda h(x) \le 0$

$\lambda \geq 0.$

(ie) $8 - 2x_1 - 3\lambda = 0$ —— (1)

$10 - 2x_2 - 2\lambda = 0$ —— (2)

$\lambda(3x_1 + 2x_2 - 6) = 0$

$3x_1 + 2x_2 - 6 \le 0$ and $\lambda \geq 0$ —— (3)

case(i) $\lambda = 0$

The above equations become

$8 - 2x_1 = 0$

$10 - 2x_2 = 0.$

$\therefore x_1 = 4, \quad x_2 = 5.$

This solution is not feasible since $3x_1 + 2x_2 \neq 6$ when $x_1 = 4$ and $x_2 = 5$.

<u>Case (ii)</u> $\lambda \neq 0$.

Then the eqns (1), (2) and (3) becomes.

$$8 - 2x_1 - 3\lambda = 0$$
$$10 - 2x_2 - 2\lambda = 0$$
$$3x_1 - 2x_2 - 6 = 0$$
$$x_1 = \frac{8 - 3\lambda}{2}, \quad x_2 = \frac{10 - 2\lambda}{2} = 5 - \lambda.$$

Substitute in (3) we get

$$3\left(\frac{8-3\lambda}{2}\right) + 2(5 - \lambda) - 6 = 0.$$
$$13\lambda - 33 = 0$$
$$\lambda = \frac{33}{13}$$
$$\text{Max } Z = 8 \times \frac{4}{13} + 10 \times \frac{33}{13} - \frac{4}{13} - 5^2.$$
$$\text{Max } Z = 21.3.$$

<u>Solution</u> $x_1 = 4/13, \quad x_2 = \frac{33}{13}, \quad \text{Max } Z = 21.3.$

6. Minimise $Z = 0.3x_1^2 - 2x_1 + 0.4x_2^2 - 2.4x_2 + 0.6x_1x_2 + 100$

subject to

$$2x_1 + x_2 \geq 4, \quad x_1, x_2 \geq 0$$

<u>Soln:</u> Let $f(x) = 0.3x_1^2 - 2x_1 + 0.4x_2^2 - 2.4x_2 + 0.6x_1x_2 + 100$

and $h(x) = 2x_1 + x_2 - 4$

Kuhn-Tucker conditions for a minimisation problem

are
$$f_j(x) - h_j(x) = 0$$
$$\lambda h(x) = 0$$
$$h(x) \geq 0.$$
$\lambda \geq 0$ with the usual notation.

Thus we have

$$0.6x_1 - 2 + 0.6x_2 - 2\lambda = 0 \quad \text{——} (1)$$
$$0.8x_1 - 2.4 + 0.6x_1 - \lambda = 0 \quad \text{——} (2)$$
$$\lambda(2x_1 + x_2 - 4) = 0 \quad \text{——} (3)$$

and $2x_1 + x_2 - 4 \geq 0$ —— (4)

$x_1, x_2 \geq 0$, $\lambda \geq 0$

Case(i) $\lambda = 0$.

(1) & (2) becomes

$$0.6x_1 + 0.6x_2 = 2$$
$$0.8x_1 + 0.6x_2 = 2.4.$$

(ie) $3x_1 + 3x_2 = 10$

$$3x_1 + 4x_2 = 12$$

$\therefore x_2 = 2$ and $x_1 = 4/3$

But $2x_1 + x_2 = 8/3 + 2 = 14/3 \geq 4$.

This solution is feasible.

Case(ii) $\lambda \neq 0$.

The Kuhn-Tucker conditions becomes

$$0.6x_1 + 0.6x_2 = 2 + 2\lambda$$
$$0.6x_1 + 0.8x_2 = \lambda + 2.4.$$
$$2x_2 = 0.4 - \lambda$$

$\lambda = 0.4 - 0.2x_2.$

From (3)

$2x_1 + x_2 - 4 = 0.$ Therefore $0.6x_1 + 0.8x_2 = 0.4 + 0.2x_2.$

$0.6x_1 + 10x_2 = 28$

(i.e) $3x_1 + 5x_2 = 14.$

But $2x_1 + x_2 = 4$

$\therefore x_1 = -6/7 < 0, \quad x_2 = \frac{40}{7}$

This solution is not feasible since $x_1 < 0$

$\therefore$ The optimal soln is

$x_1 = 4/3, \quad x_2 = 2.$

$Z_{min} = 0.3 \times \frac{16}{9} - 8/3 + 0.4 \times 4 - 2.4 \times 2 + 0.6 \times \frac{4}{3} \times 2 + 100$

$= \frac{292}{3} = 97.33.$

Soln is $x_1 = 4/3, \quad x_2 = 2, \quad Z_{min} = 97.33.$

7. Using Lagrangean Multiplier Maximise

$Z = 10x_1 + 4x_2 + 4x_1x_2 - x_1^2 - 5x_2^2$

Soln: Rewriting the problem.

Max $Z = 10x_1 + 4x_2 + 4x_1x_2 - x_1^2 - 5x_2^2.$

Subject to $x_1 + x_2 - 6 = 0$

$x_1, x_2 \geq 0$

The Lagrangean function can be taken as

$L = 10x_1 + 4x_2 + 4x_1x_2 - x_1^2 - 5x_2^2 - \lambda(x_1 + x_2 - 6).$

The necessary conditions for extrema is given by

$$\frac{\partial L}{\partial x_1} = 10 - 2x_1 + 4x_2 - \lambda = 0 \quad \text{———} (1)$$

$$\frac{\partial L}{\partial x_2} = 4 + 4x_1 - 10x_2 - \lambda = 0. \quad \text{———} (2)$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 6) = 0 \quad \text{———} (3)$$

From (1) and (3)

$$10 - 2x_1 + 4x_2 = \lambda \quad \text{———} (4)$$

$$4 + 4x_1 - 10x_2 = \lambda \quad \text{———} (5)$$

$$x_1 + x_2 = 6 \quad \text{———} (6)$$

(4) = (5) $\Rightarrow$ $10 - 2x_1 + 4x_2 = 4 + 4x_1 - 10x_2$

$$3x_1 - 7x_2 = 3. \quad \text{———} (7)$$

$3 \times (6) \Rightarrow$ $3x_1 + 3x_2 = 18$ ——— (8)

(8) − (9) $\Rightarrow$ $10x_2 = 15$

$$x_2 = 3/2.$$

$$x_1 = 6 - 3/2 = 9/2.$$

Soln is $x_1 = 9/2$, $x_2 = 3/2$.

$$\text{Max } Z = 10 \times 9/2 + 4 \times 3/2 - 81/4 + 4 \times 27/4 - 5 \times 9/4.$$

$$= 78 - 63/2 = 93/2.$$

Now $n = 2$

$$\Delta_{n+1} = \Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 4 \\ 1 & 4 & -10 \end{vmatrix} = 20 > 0$$

$\therefore x_1 = 9/2$, $x_2 = 3/2$ maximises Z

$$\frac{\partial^2 Z}{\partial x_1^2} = -20 < 0$$

$\therefore$ $x_1 = 9/2$, $x_2 = 3/2$ Maximises Z

Max $Z = 93/2$.

8. Determine the maximum and minimum value of the function $f(x) = (3x-4)^2(2x-3)^2$

Soln:

$$f(x) = (3x-4)^2(2x-3)^2.$$

$$f'(x) = 6(3x-4)(2x-3)^2 + 4(2x-3)(3x-4)^2.$$

$$= 2(2x-3)(3x-4)(24x-34)$$

Stationary values are given by

$$f'(x) = 0$$

(ie) $(2x-3)(3x-4)(24x-34) = 0$

$$x = 3/2,\ 4/5,\ 17/12.$$

$$f''(x) = 2(3x-4)(24x-34) + 3(2x-3)(24x-34) + 24(2x-3)(3x-4).$$

Case (1) $x = 3/2$.

$$f''(3/2) = 2(9/2 - 4)(36-34)$$

$$= 2 > 0$$

$x = 3/2$ Minimises $f(x)$.

Min $f(x) = 0$

Case(ii) Consider $x = \frac{4}{3}$

$$f''\left(\frac{4}{3}\right) = 3\left(\frac{8}{3} - 3\right)(32 - 34) = 2 > 0.$$

$\therefore x = \frac{4}{3}$ minimises $f(x)$

Minimum $f(x) = f(4/3) = 0$

Case(iii) Consider $x = \frac{17}{12}$.

$$f''\left(\frac{17}{12}\right) = 24\left(\frac{34}{12} - 3\right)\left(\frac{51}{12} - 4\right) < 0$$

$x = \frac{17}{12}$ maximises $f(x)$.

$$\text{Min } f(x) = f(17/12) = \left(\frac{51}{12} - 4\right)^2 \left(\frac{34}{12} - 3\right)^2 = \frac{1}{576}$$

9 Maximize $Z = 7x_1^2 + 6x_1 + 5x_2$.

Subject to $x_1 + 2x_2 \le 10$

$x_1 - 3x_2 \le 9$.

$x_1, x_2 \ge 0$

Soln: Let $f(x) = 7x_1^2 + 6x_1 + 5x_2$

$h^1(x) = x_1 + 2x_2 - 10$

$h^2(x) = x_1 - 3x_2 - 9$

The Kuhn-Tucker conditions for the maximization position become

$$f_j(x) - \sum_{i=1}^{2} \lambda_i h^i(x) = 0.$$

$$\lambda_1 h^1(x) = 0$$

$$\lambda_2 h^2(x) = 0$$

$h^i(x) \le 0 \quad i = 1, 2$

$\lambda_1, \lambda_2 \ge 0, \quad x_1, x_2 \ge 0$

Thus $14x_1 + 6 - \lambda_1 = 0$ —— (1)

$5 - 2\lambda_1 + 3\lambda_2 = 0$ —— (2)

$\lambda_1(x_1 + 2x_2 - 10) = 0$ —— (3)

$\lambda_2(x_1 - 3x_2 - 9) = 0$ —— (4)

$x_1 + 2x_2 - 10 \le 0.$ —— (5)

$x_1 - 3x_2 - 9 \le 0$ —— (6)

$x_1, x_2 \ge 0, \quad \lambda_1, \lambda_2 \ge 0$

Case (i) $\lambda_1 = 0, \quad \lambda_2 = 0.$

(2) gives absurd result namely $5 = 0$. No feasible soln in this case.

Case (ii) $\lambda_1 = 0, \quad \lambda_2 \ne 0$

(1) $\Rightarrow$ $14x_1 + 6 = 0$

$x_1 = -6/14 < 0.$

No feasible in this case

Case (iii) $\lambda_1 \ne 0, \quad \lambda_2 = 0$

$14 \cdot x_1 + 6 - \lambda_1 = 0$

$5 - 2\lambda_1 = 0$

$x_1 + 2x_2 - 10 = 0$

$\lambda_1 = 14x_1 + 6$

$\therefore \quad 5 - 2(14x_1 + 6) = 0$

$x_1 = -7/28 < 0$

No feasible soln.

case(iv) $\lambda_1 \neq 0, \quad \lambda_2 \neq 0.$

$\therefore$ (3) & (4) $\Rightarrow$ $x_1 + 2x_2 - 10 = 0$

$x_1 - 3x_2 - 9 = 0$

$5x_2 - 1 = 0, \quad x_2 = 1/5$

$x_1 = 10 - 2x_2 = \frac{48}{5}; \quad \lambda_1 = 14x_1 + 6 = \frac{702}{5}$

$\lambda_2 = \frac{1399}{15}$

$\text{Max } Z = 7\left(48/5\right)^2 + 6 \times 48/3 + 1 = 703.72.$

10. Find the stationary points of $f(x) = 4x^4 - x^2 + 5$ and determine the nature of the stationary points.

Soln

$f(x) = 4x^4 - x^2 + 5$

$f'(x) = 16x^3 - 2x.$

Stationary points are

$f'(x) = 0$

$16x^3 - 2x = 0$

$2x(8x^2 - 1) = 0$

$x = 0, \quad 8x^2 - 1 = 0$

$x = \pm \frac{1}{2\sqrt{2}}.$

$f''(x) = 48x^2 - 2.$

Case (i) Consider $x = 0$

$f''(0) = -2 < 0.$

$x = 0$ maximises $f(x)$

and the maximum value of $f(x) = 5$

Case (ii) Consider $x = \frac{1}{2\sqrt{2}}$.

$$f'' = 48 \times \frac{1}{8} - 2 = 4 > 0$$

$x = \frac{1}{2\sqrt{2}}$ minimises $f(x)$ and the minimum value of $f(x)$ is

$$f\left(\frac{1}{2\sqrt{2}}\right) = 4 \times \frac{1}{64} - \frac{1}{8} + 5 = \frac{79}{16}$$

Case (iii) Consider $x = -\frac{1}{2\sqrt{2}}$.

$$f''\left(-\frac{1}{2\sqrt{2}}\right) = 48 \times \frac{1}{8} - 2 = 4 > 0$$

$\therefore x = \frac{-1}{2\sqrt{2}}$ also minimises $f(x)$ and the minimum value of $f(x) = \frac{79}{16}$

UNIT – II TRANSPORTATION MODEL.

PART B Questions

1. Find the starting solution of the following transportation model.

1	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

using (i) North West corner Method.

(ii). Least cost method.

(iii) Vogel's approximation method.

Soln: Since $\Sigma a_i = \Sigma b_j = 30$, the given transportation problem is balanced. Hence there exists a basic feasible soln to this problem

(i). North west Corner Rule.

1 (7)	2	6	7
0 (3)	4 (9)	2	~~12~~ 9
3	1 (1)	5 (10)	~~11~~ 10
~~10~~ 3	~~10~~ 1	10	

∴ The initial transportation cost

$= Rs\ 1\times7 + 0\times3 + 4\times9 + 1\times1 + 5\times10$

$= Rs\ 94$

Least cost Method :

1	2	6 (7)	~~7~~
0 (10)	4	2 (2)	~~12~~ 2
3	1 (10)	5 (1)	~~11~~ 1
10	10	~~10~~ ~~8~~ 7	

The initial transportation cost $= 6\times7 + 0\times10 + 2\times2 + 1\times10 + 5\times1$

$= 61.$

Vogel's Approximation Method.

1 (7)	2	6	7 (1) (1)
0 (2)	4	2 (10)	~~12~~ 2 (2) (4) (1)
3 (1)	1 (10)	5	~~11~~ 1 (2) (2) (2)
~~10~~ ~~8~~ 1 (1) (1) (2)	10 (1) (1) (1)	10 (3)	

∴ The initial transportation cost

$= 1\times 7 + 0\times 2 + 2\times 10 + 3\times 1 + 1\times 10$

$= 40$

2. Find The optimal transportation problem of the following matrix using least cost method for finding the critical solution.

Market

		A	B	C	D	E	
Factory	P	4	1	2	6	9	100
	Q	6	4	3	5	7	120
	R	5	2	6	4	8	120
Demand		40	50	70	90	90	

Soln:

Since $\sum a_i = \sum b_j = 340$, the given transportation problem is balanced. Therefore there exists a basic feasible solution to this problem.

By using least cost method, the initial solution is as shown in the following table.

4	1 (50)	2 (50)	6	9
6 (10)	4	3 (20)	5	7 (90)
5 (30)	2	6	4 (90)	8

The initial transportation cost $= 1\times 50 + 2\times 50 + 6\times 10 + 3\times 20 + 7\times 90 + 5\times 30 + 4\times 90 = 1410$

For optimality : Since the number of non-negative independent allocations is $(m+n-1)$ we

4 5 -1	1 50	2 50	6 4 2	9 6 3	$u_1 = -1$
6 10	4 2 2	3 20	5 5 0	7 90	$u_2 = 0$
5 30	2 1 1	6 2 4	4 90	8 6 2	$u_3 = -1$
$v_1=6$	$v_2=2$	$v_3=3$	$v_4=5$	$v_5=7$.	

Since $d_{11} = -1 < 0$, the current soln is not optimal.

Here (1,1) having the negative value $d_{11} = -1$. We draw a closed loop containing of horizontal and vertical lines beginning and ending at this cell (1,1) and having its other corners at some occupied cells. Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners we have

4 $+\theta$	1 50	2 50 $-\theta$	6	9
6 10 $-\theta$	4	3 20 $+\theta$	5	7
5 30	2	6	4 90	8

From the two cells (1,3), (2,1) having $-\theta$, we find

That the minimum of the allocations 5, 10 is 10.
Add this 10 to the cells with $+\theta$ and subtract this 10 to the cells with $-\theta$.

4 10	1 50	2 40	6	9
6	4	3 30	5	7 90
5 30	2	6	4 90	8

Apply Modi method

4 10	1 50	2 40	6 3 (3)	9 6 (3)	$u_1 = 0$
6 5 (1)	4 2 (2)	3 30	5 4 (1)	7 90	$u_2 = 1$
5 30	2 2 (0)	6 3 (3)	4 90	8 7 (1)	$u_3 = 1$
$v_1 = 4$	$v_2 = 1$	$v_3 = 2$	$v_4 = 3$	$v_5 = 6$	

Since all $d_{ij} > 0$ with $d_{32} = 0$. The current soln is optimal.

Transportation cost $= 4 \times 10 + 1 \times 50 + 2 \times 40 + 3 \times 30 + 7 \times 90 + 5 \times 30 + 4 \times 90$

$= $ Rs. 1400

3. Solve the following transportation problem to minimize the total cost of transportation.

Destination

		1	2	3	4	Supply
Origin	1	14	56	48	27	70
	2	82	35	21	81	47
	3	99	31	71	63	93
Demand		70	35	45	60	

Soln: Since $\sum a_i = \sum b_j = 210$, The given transportation problem is balanced. Therefore There exists a basic feasible soln to this problem.

By using vogel's approximation method we find initial soln.

14 (70)	56	48	27
82	35	21 (45)	81 (2)
99	31 (35)	71	63 (58)

Since $m+n-1 = 6$. this basic feasible soln is degenerate.

To resolve degeneracy, we allocate a very small quantity ϵ to The cell (1,4) so That the number of occupied cells becomes $m+n-1$.

14 70	56	48	27 ϵ
82	35	21 45	81 2
99	31 35	71	63 58

To find the optimal soln:

14 70	56 −5 61	48 −33 81	27 ϵ	$u_1 = 27$
82 68 14	35 49 −14	21 45	81 2	$u_2 = 81$
99 50 49	31 35	71 3 68	63 58	$u_3 = 63$
$v_1 = -13$	$v_2 = -32$	$v_3 = -60$	$v_4 = 0$	

Since $d_{22} = -14 < 0$, the soln under the test is not optimal.

14 70	56	48	27 ϵ
82	35 $+\theta$	21 45	81 2 $-\theta$
99	31 35 $-\theta$	71	63 58 $+\theta$

From the two cells (2,4), (3,2) having $-\theta$, we find that the minimum of the allocation {2, 35} is 2. Add this 2

to the cells with $+\theta$ and subtract this 2 to the cells with $-\theta$.

14 70	56	48	27 ϵ
82	35 2	21 45	81
99	31 33	71	63 60

We apply modi method for optimality.

14 70	56 −5 61	48 −19 67	27 ϵ	$u_1 = -40$
82 54 28	35 2	21 45	81 67 14	$u_2 = 0$
99 50 49	31 33	71 17 54	63 60	$u_3 = -4$
$v_1 = 54$	$v_2 = 35$	$v_3 = 21$	$v_4 = 67$	

Since all $d_{ij} > 0$, the solution under the test is optimal.

$\therefore$ The transportation cost $= 14 \times 70 + 27 \times \epsilon + 35 \times 2 + 21 \times 45$
$+ 31 \times 33 + 63 \times 60$
$= 6798$ as $\epsilon \to 0$

4 (a) A batch of 4 jobs can be assigned to 5 different machines. The set up time for each job on various machines is given below.

Machine

	1	2	3	4	5
1	10	11	4	2	8
2	7	11	10	14	12
3	5	6	9	12	14
4	13	15	11	10	7.

Find an optimal assignment of jobs to machines which will minimize the total setup time.

Soln:

The matrix of the given pbm.

$$\begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 15 & 11 & 10 & 7 \end{pmatrix}$$

Since the number of rows is less than the number of columns. The gn pbm is unbalanced.

The balanced cost matrix.

$$\begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 15 & 11 & 10 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Select the smallest cost element in each row (column) ~~from~~ and subtract this from all the elements of the corresponding row (column)

$$\begin{pmatrix} 8 & 9 & 2 & 0 & 6 \\ 0 & 4 & 3 & 7 & 5 \\ 0 & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since each row and each column contains atleast one ~~row~~ zero we shall make the assignment in rows & columns.

$$\begin{pmatrix} 8 & 9 & 2 & (0) & 6 \\ (0) & 4 & 3 & 7 & 5 \\ \cancel{0} & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & (0) \\ \cancel{0} & (0) & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix}$$

Since there are some rows and columns without assignment the current assignment is not optimal.

Cover all zeros by drawing a minimum number of straight lines.

$$\begin{pmatrix} 8 & 9 & 2 & 0 & 6 \\ 0 & 4 & 3 & 7 & 5 \\ 0 & (1) & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Subtract 1 from all the uncovered elements, add ~~to~~ this 1 to those elements which lie in the intersection of these st lines and do not change the remaining elts which lie on the st lines

$$\begin{pmatrix} 9 & 9 & 2 & 0 & 6 \\ 0 & 3 & 2 & 6 & 4 \\ 0 & 0 & 3 & 6 & 8 \\ 7 & 8 & 4 & 3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 9 & 2 & (0) & 6 \\ (0) & 3 & 2 & 6 & 4 \\ \times & (0) & 3 & 6 & 8 \\ 7 & 8 & 4 & 3 & (0) \\ 1 & \times & (0) & \times & \times \end{pmatrix}$$

The optimum total set up time $= 2+7+6+7$
$= 22$ hrs.

4(b) A machine shop purchased a drilling machine and two lathes of different capacities. The positioning of the machines among 4 possible locations on the shop floors is important from the standard of materials handling. Given the cost estimate per unit time of materials below determine the optimal location of the machines.

Location.

	1	2	3	4
Lathe 1.	12	9	12	9
Drill	15	not suitable	13	20
Lathe 2.	4	8	10	6

Soln: Since the drilling machine is not suitable for location 2 the corresponding cost element should be taken as ∞. Thus the cost matrix is

$$\begin{pmatrix} 12 & 9 & 12 & 9 \\ 15 & \infty & 13 & 20 \\ 4 & 8 & 10 & 6 \end{pmatrix}$$

Since the number of rows is less than the number of

Columns, we add a dummy row with zero cost elements.

$$\begin{pmatrix} 12 & 9 & 12 & 9 \\ 15 & \infty & 13 & 20 \\ 4 & 8 & 10 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Select the smallest cost in each row and subtract this from all the cost elements of the corresponding row. We get the reduced matrix

$$\begin{pmatrix} 3 & 0 & 3 & 0 \\ 2 & \infty & 0 & 7 \\ 0 & 4 & 6 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since each row and each column contains atleast one zero we shall make the assignment in rows and columns having single zero. We get

$$\begin{pmatrix} 3 & (0) & 3 & \cancel{0} \\ 2 & \infty & (0) & 7 \\ (0) & 4 & 6 & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} & (0) \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

∴ The optimum assignment schedule is

Lathe 1 → Location 2, Drill → Location 3

Lathe 2 → Location 1 Dummy drill → Location 4.

and the optimum assignment cost $= 9 + 13 + 4 + 0$

$= 26$ units of cost.

5(a) Solve the following travelling salesman problem so as to minimize the cost per cycle.

From \ To	A	B	C	D	E
A	—	3	6	2	3
B	3	—	5	2	3
C	6	5	—	6	4
D	2	2	6	—	6
E	3	3	4	6	—

Soln:

The cost matrix of the given travelling salesman problem is

$$\begin{pmatrix} \infty & 3 & 6 & 2 & 3 \\ 3 & \infty & 5 & 2 & 3 \\ 6 & 5 & \infty & 6 & 4 \\ 2 & 2 & 6 & \infty & 6 \\ 3 & 3 & 4 & 6 & \infty \end{pmatrix}$$

Subtract the smallest cost element in each row from all the elements of the corresponding row we get.

$$\begin{pmatrix} \infty & 1 & 4 & 0 & 1 \\ 1 & \infty & 3 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 4 & \infty & 4 \\ 0 & 0 & 1 & 3 & \infty \end{pmatrix}$$

Subtract the smallest cost element in each column from all the elements of the corresponding column. We get

$$\begin{pmatrix} \infty & 1 & 3 & 0 & 1 \\ 1 & \infty & 2 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 3 & \infty & 4 \\ 0 & 0 & 0 & 3 & \infty \end{pmatrix}$$

Now we shall make the assignment in rows and columns having single zeroes we get

$$\begin{pmatrix} \infty & 1 & 3 & (0) & 1 \\ 1 & \infty & 2 & \cancel{0} & 1 \\ 2 & 1 & \infty & 2 & (0) \\ \cancel{0} & (0) & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & (0) & 3 & \infty \end{pmatrix}$$

Since some rows and columns are without assignment, the current assignment is not optimal. Cover all the zeros by drawing a minimum number of straight lines

$$\begin{pmatrix} \infty & (1) & 3 & 0 & 1 \\ 1 & \infty & 2 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 3 & \infty & 4 \\ 0 & 0 & 0 & 3 & \infty \end{pmatrix}.$$

Subtract the smallest uncovered cost element 1 from all uncovered elements add this 1 to those elements which are in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines.

We have

$$\begin{pmatrix} \infty & 0 & 2 & 0 & 0 \\ 0 & \infty & 1 & 0 & 0 \\ 2 & 1 & \infty & 3 & 0 \\ 0 & 0 & 3 & \infty & 4 \\ 0 & 0 & 0 & 4 & \infty \end{pmatrix},$$

We shall make the assignment in rows and columns having single zero we get

$$\begin{pmatrix} \infty & \cancel{0} & 2 & (0) & \cancel{0} \\ (0) & \infty & 1 & \cancel{0} & \cancel{0} \\ 2 & 1 & \infty & 3 & (0) \\ \cancel{0} & (0) & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & (0) & 4 & \infty \end{pmatrix}$$

Since each row and each column contains exactly one encircled zero, the current assignment is optimal.

∴ The optimal assignment schedule is given by

A → D, B → A, C → E, D → B, E → C

(ie) A → D → B → A, C → E → C.

and the corresponding optimum assignment cost

$= 2+3+4+2+4$

$= 15$ units of cost.

But this problem does not satisfy the route condition, we start with making an assignment at (2,3) instead of zero assignment at (2,1). The resulting feasible solution will then be

$$\begin{pmatrix} \infty & \cancel{0} & 2 & (0) & \cancel{0} \\ \cancel{0} & \infty & (1) & \cancel{0} & \cancel{0} \\ 2 & 1 & \infty & 3 & (0) \\ \cancel{0} & (0) & 3 & \infty & 4 \\ (0) & \cancel{0} & \cancel{0} & 4 & \infty \end{pmatrix}.$$

The optimum assignment is given by

(ie) A → D → B → C → E → A.

Also when an assignment is made at (3,2) instead

of zero assignment at (3, 5), the resulting feasible solution will then be

$$\begin{pmatrix} \infty & \cancel{0} & 2 & (0) & \cancel{0} \\ \cancel{0} & \infty & (1) & \cancel{0} & \cancel{0} \\ 2 & 1 & \infty & 3 & (0) \\ \cancel{0} & (0) & 3 & \infty & 4 \\ (0) & \cancel{0} & \cancel{0} & 4 & \infty \end{pmatrix}$$

The optimum assignment is given by

$A \to D \to B \to C \to E \to A$.

Also when an assignment is made at (3, 2) instead of zero assignment at (3, 5), the resulting feasible solution will be

$$\begin{pmatrix} \infty & \cancel{0} & 2 & \cancel{0} & (0) \\ \cancel{0} & \infty & 1 & (0) & \cancel{0} \\ 2 & (1) & \infty & 3 & \cancel{0} \\ (0) & \cancel{0} & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & (0) & 4 & \infty \end{pmatrix}$$

The optimum assignment is given by

$A \to E \to C \to B \to D \to A$.

∴ For the given travelling salesman problem, the optimum assignment schedule is given by

$A \to D \to B \to C \to E \to A$ (or)

$A \to E \to C \to B \to D \to A$

5. (b) Solve the assignment problem for maximization given the profit matrix.

Machines

Job	P	Q	R	S
A	51	53	54	50
B	47	50	48	50
C	49	50	60	61
D	63	64	60	60

Soln: The profit matrix is

$$\begin{pmatrix} 51 & 53 & 54 & 50 \\ 47 & 50 & 48 & 50 \\ 49 & 50 & 60 & 61 \\ 63 & (64) & 60 & 60 \end{pmatrix}$$

Since this is a maximization problem it can be converted into an equivalent minimization problem by subtracting all the profit elements in the profit matrix from the highest profit element 64 of this profit matrix.

$$\begin{pmatrix} 13 & 11 & 10 & 14 \\ 17 & 14 & 16 & 14 \\ 15 & 14 & 4 & 3 \\ 1 & 0 & 4 & 4 \end{pmatrix}$$

Select the smallest cost in each row and subtract this from all the cost elements of the corresponding row. We get.

$$\begin{pmatrix} 3 & 1 & 0 & 4 \\ 3 & 0 & 2 & 0 \\ 12 & 11 & 1 & 0 \\ 1 & 0 & 4 & 4 \end{pmatrix}$$

Select the smallest cost element in each column and subtract this from all the cost elements of the corresponding column we get

$$\begin{pmatrix} 2 & 1 & 0 & 4 \\ 2 & 0 & 2 & 0 \\ 11 & 11 & 1 & 0 \\ 0 & 0 & 4 & 4 \end{pmatrix}.$$

Since each row and each column contains atleast one zero we shall make the assignment in rows

and columns having single zero. We get

$$\begin{pmatrix} 2 & 1 & (0) & 4 \\ 2 & (0) & 2 & \cancel{0} \\ 11 & 11 & 1 & (0) \\ (0) & \cancel{0} & 4 & 4 \end{pmatrix}$$

Since each row and ~~row~~ each column contains exactly one encircled zero the current assignment is optimal.

$\therefore$ The optimum assignment schedule

$= Rs\ (54 + 50 + 61 + 63)$

$= Rs\ 228.$

UNIT IV

CLASSICAL OPTIMISATION THEORY.

PART-B

1. Investigate the functions $f(x) = 6x^5 - 43x^3 + 12$ for maxima and minima.

Soln:

$$f(x) = 6x^5 - 4x^3 + 12 \quad \text{——— (1)}$$

$$f'(x) = 30x^4 - 12x^2 \quad \text{——— (2)}$$

At an extremum. $f'(x) = 0$

$f'(x) = 0$ gives $30x^4 - 12x^2 = 0$

$$3x^2(10x^2 - 4) = 0$$

$$x = 0, \quad 10x^2 - 4 = 0$$

$$(i.e) \quad x = \pm 2/\sqrt{10}$$

$\therefore$ $x = 0$, $x = 2/\sqrt{10}$ and $x = -2/\sqrt{10}$ are the stationary values.

To determine the nature of these From (2)

$$f''(x) = 120x^3 - 24x.$$

Case (i) $x = 0$

$$f''(0) = 0$$

$$f'''(0) = -24 \neq 0$$

$\therefore$ $x = 0$ gives a point of inflexion ie, $(0,0)$ in a point of inflexion for the function $f(x)$.

Case (ii) Consider $x = \frac{2}{\sqrt{10}}$

$$f''\left(\frac{2}{\sqrt{10}}\right) = 48/\sqrt{10}\left(5 \times 4/10 - 1\right) = 48/\sqrt{10} > 0$$

$\therefore$ $x = \frac{2}{\sqrt{10}}$ minimise $f(x)$.

1. Using Gomory's cutting plane method

Maximize $Z = 2x_1 + 2x_2$.

Subject to $5x_1 + 3x_2 \le 8$

$2x_1 + 4x_2 \le 8$

and $x_1, x_2 \ge 0$ and are all integers.

Soln: Maximize $Z = 2x_1 + 2x_2 + 0x_3 + 0x_4$.

Subject to $5x_1 + 3x_2 + x_3 + 0x_4 = 8$

$2x_1 + 4x_2 + 0x_3 + x_4 = 8$

and $x_1, x_2, x_3, x_4 \ge 0$

The initial basic feasible solution is given by

$x_3 = 8, \quad x_4 = 8$.

Initial iteration:

		c_j	2	2	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	8	(5)	3	1	0	$8/5$ *
0	x_4	8	2	4	0	1	$8/2$
	$Z_j - C_j$	0	-2	-2	0	0	

First iteration: Introduce x_1 and drop x_3

		c_j	2	2	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
2	x_1	$8/5$	1	$3/5$	$1/5$	0	$8/3$
0	x_4	$24/5$	0	$14/5$	$-2/5$	1	$\frac{12}{7}$ *
	$Z_j - C_j$	$16/5$	0	$-4/5$	$2/5$	0	

Second iteration:

Introduce x_2 and drop x_4.

		c_j	2	2	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4
2	x_1	$4/7$	1	0	$2/7$	$-3/14$
2	x_2	$12/7$	0	1	$-1/7$	$5/14$
$Z_j - c_j$		$32/7$	0	0	$2/7$	$2/7$

Since all $Z_j - c_j \geq 0$ the current basic feasible soln is optimal but non-integer.

To obtain the optimum integer soln, we have to construct a fractional cut constraint.

$$x_1 = \frac{4}{7} = 0 + 4/7 = [x_{B1}] + f_1$$

$$x_2 = \frac{12}{7} = 1 + 5/7 = [x_{B2}] + f_2.$$

$\therefore$ Max $\{f_1, f_2\}$ = Max $\left\{\frac{4}{7}, 5/7\right\} = 5/7$, which corresponds to the second row.

$$\frac{12}{7} = x_2 - \frac{1}{7}x_3 + \frac{5}{14}x_4.$$

$$\text{(ie)} \quad 1 + \frac{5}{7} = x_2 + (-1 + 6/7)x_3 + 5/14\, x_4.$$

The fractional cut constraint is given by

$$6/7\, x_3 + \frac{5}{14}x_4 \geq 5/7.$$

$$-6/7\, x_3 - 5/14\, x_4 \leq -5/7.$$

$$-6/7\, x_3 - 5/14\, x_4 + s_1 = -5/7.$$

s_1 – Gomorian slack.

$x_3 = 5/6 = 0 + 5/6 = [X_{B3}] + f_3$

$\therefore \text{Max}\{f_1, f_2, f_3\} = \text{Max}\{\frac{1}{3}, 5/6, 5/6\} = 5/6$

$\frac{11}{6} = x_2 + 5/12\, x_4 - 1/6\, s_1$

$1 + 5/6 = x_2 + 5/12\, x_4 + (-1 + 5/6)\, s_1.$

The fractional cut construct is given by

$5/12\, x_4 + 5/6\, x_1 \geq 5/6$

$-5/12\, x_4 - 5/6\, s_1 \leq -5/6$

$-5/12\, x_4 - 5/6\, s_1 + s_2 = -5/6$

where s_2 is the Gomorian slack.

		c_j	2	2	0	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2
2	x_1	$1/3$	1	0	0	$-1/3$	$1/3$	0
2	x_2	$11/6$	0	1	0	$5/12$	$-1/6$	0
0	x_3	$5/6$	0	0	1	$5/12$	$-7/6$	0
0	s_2	$-5/6$	0	0	0	$-5/12$	$-5/6$	1
	$Z_j - C_j$	$\frac{13}{3}$	0	0	0	$1/6$	$1/3$	0

Since $s_2 = -5/6$, s_2 leaves the basis

Also $\text{Max}\left\{\frac{Z_j - C_j}{a_{ik}}, a_{ik} < 0\right\}$

$= \text{Max}\left\{\frac{\frac{1}{6}}{-\frac{-5}{12}}, \frac{\frac{1}{3}}{-\,-5/6}\right\} = \text{Max}\{-2/5, -2/5\}$

$= -2/5$

	c_j		2	2	0	0	0
C_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1
2	x_1	$4/7$	1	0	$2/7$	$-3/4$	0
2	x_2	$12/7$	0	1	$-1/7$	$5/14$	0
0	s_1	$-5/7$	0	0	$-6/7$	$-5/14$	1
	$z_j - c_j$	$\frac{32}{7}$	0	0	$2/7$	$2/7$	0

Since $s = -5/7$, s_1 leaves the basis

Also $\text{Max}\left\{ \frac{z_j - c_j}{a_{ik}},\ a_{ik} < 0 \right\} = \text{Max}\left\{ \frac{\frac{2}{7}}{-6/7},\ \frac{2/7}{\frac{-5}{14}} \right\}$

$= \text{Max}\left\{ -\frac{1}{3}, -4/5 \right\} = -1/3$ which corresponds to x_3

Third iteration:

	c_j		2	2	0	0	0
C_B	y_B	x_B	x_1	x_2	x_3	x_4	s_1
2	x_1	$\frac{1}{3}$	1	0	0	$-1/3$	$1/3$
2	x_2	$\frac{11}{6}$	0	1	0	$5/12$	$-1/6$
0	x_3	$5/6$	0	0	1	$5/12$	$-7/6$
	$z_j - c_j$	$\frac{13}{5}$	0	0	0	$1/6$	$1/3$

Since all $z_j - c_j \geq 0$ and all $x_{Bi} \geq 0$ the current soln is feasible and optimal but non integer.

$\text{Max } Z = 2,\ x_1 = 0,\ x_2 = 2.$

$x_1 = 1/3 = 0 + 1/3 = [x_{B1}] + f_1$

$x_2 = \frac{11}{6} = 1 + 5/6 = [x_{B2}] + f_2$,

		C_j	2	2	0	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2
2	x_1	1	1	0	0	0	1	$-4/5$
2	x_2	1	0	1	0	0	-1	1
0	x_3	0	0	0	1	0	-2	1
0	x_4	2	0	0	0	1	2	$-\frac{12}{5}$
$Z_j - C_j$		4	0	0	0	0	0	$2/5$

Since all $Z_j - C_j \geq 0$ and all $X_{Bi} \geq 0$ the current soln is feasible integer optimal.

$$\text{Max } Z = 4, \quad x_1 = 1, \quad x_2 = 1.$$

2. Solve the following Mixed integer Programming by Gomory's cutting plane algorithm.

$$\text{Max } Z = x_1 + x_2.$$

Subject to $3x_1 + 2x_2 \leq 5$

$x_2 \leq 5$

and $x_1, x_2 \geq 0$ and x_1 an integer.

Soln:

$$\text{Max } Z = x_1 + x_2 + 0x_3 + 0x_4.$$

Subject to $3x_1 + 2x_2 + x_3 + 0x_4 = 5$

$0x_1 + x_2 + 0x_3 + x_4 = 2$

and $x_1, x_2, x_3, x_4 \geq 0$.

The initial basic feasible soln is

$$x_3 = 5, \quad x_4 = 2.$$

First iteration

		C_j	1	1	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
0	x_3	5	(3)	2	1	0	$5/3$
0	x_4	2	0	1	0	1	—
$Z_j - C_j$		0	-1	-1	0	0	

First iteration

Introduce x_1 and drop x_3

	c_j		1	1	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ
1	x_1	$5/3$	1	$2/3$	$1/3$	0	$\frac{5}{2}$
0	x_4	2	0	(1)	0	1	2
	$Z_j - C_j$	$5/3$	0	$-1/3$	$1/3$	0	

Second iteration Introduce x_2 and drop x_4.

	c_j		1	1	0	0
C_B	X_B	Y_B	x_1	x_2	x_3	x_4
1	x_1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	$-2/3$
1	x_2	2	0	1	0	1
	$Z_j - C_j$	$7/3$	0	0	$\frac{1}{3}$	$\frac{1}{3}$

Here x_1 is non integer.

From the 1^{st} row

$$\frac{1}{3} = x_1 + 0x_2 + \frac{1}{3}x_3 - 2/3\, x_4.$$

$\therefore$ the Gomorian constraint is given by

$$\frac{1}{3}x_3 + \left(\frac{\frac{1}{3}}{\frac{1}{3}-1}\right)\left(-2/3\right)x_4 \geq 1/3.$$

$$\frac{1}{3}x_3 + 1/3\,x_4 \geq 1/3 \Rightarrow -1/3\,x_3 - 1/3\,x_4 + S_1 = -1/3$$

(3) Use Branch and Bound Method to solve the following IPP.

Max $Z = 2x_1 + 2x_2$

Subject to $5x_1 + 3x_2 \le 8$

$x_1 + 2x_2 \le 4$,

and $x_1, x_2 \ge 0$ and integer.

Soln: Max $Z = 2x_1 + 2x_2 + 0x_3 + 0x_4$.

Subject to $5x_1 + 3x_2 + x_3 + 0x_4 = 8$

$x_1 + 2x_2 + 0x_3 + x_4 = 4$

$x_1, x_2, x_3, x_4 \ge 0$.

Initial iteration

		C_j	2	2	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ.
0	$\cancel{8}\, x_3$	8	(5)	3	1	0	8/5
0	x_4	4	1	2	0	1	4
$Z_j - C_j$		0	-2	-2	0	0	

First iteration Introduce x_1 and drop x_3

		C_j	2	2	0	0	
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	θ.
2	x_1	8/5	1	3/5	1/5	0	8/3
0	x_4	12/5	0	7/5	$-1/5$	1.	$\frac{12}{7}$.
$Z_j - C_j$		16/5	0	$-4/5$	2/5	0	

	c_j		1	1	0	0	0
C_B	X_B	Y_B	x_1	x_2	x_3	x_4	S_1
1	x_1	$1/3$	1	0	$1/3$	$-2/3$	0
1	x_2	2	0	1	0	1	0
0	S_1	$-1/3$	0	0	$-1/3$	$-1/3$	1
$Z_j - c_j$		$7/3$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0

Here the soln is optimal but infeasible.

So we use dual simplex method.

Since $S_1 = -1/3$, S_1 leaves the basis

$$\text{Max}\left\{\frac{Z_j - c_j}{a_{ik}}, a_{ik} < 0\right\} = \text{Max}\left\{\frac{1/3}{-1/3}, \frac{1/3}{-1/3}\right\}.$$

$$= \text{Max}\{-1, -1\} = -1.$$

<u>Third iteration</u> Drop S_1 and introduce x_3

	c_j		1	1	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	x_4	S_1
1	x_1	0	1	0	0	-1	1
1	x_2	2	0	1	0	1	0
0	x_3	1	0	0	1	1	-3
$Z_j - c_j$		2	0	0	0	0	1

Since all $Z_j - c_j \geq 0$ and all $X_{Bi} \geq 0$ the current soln is feasible and optimal.

$\therefore$ Max $Z = 2$, $x_1 = 0$, $x_2 = 2$.

Second iteration :

Introduce x_2 and drop x_4.

	c_j		2	2	0	0.
C_B	Y_B	X_B	x_1	x_2	x_3	x_4
2	x_1	$4/7$	1	0	$2/7$	$-3/7$
2	x_2	$12/7$	0	1	$-1/7$	$5/7$
$Z_j - c_j$		$32/7$	0	0	$2/7$	$4/7$

Since all $Z_j - c_j \geq 0$ the soln is optimal but noninteger.

$$\text{Max } Z = \frac{32}{7}$$

$$x_1 = 4/7$$

$$x_2 = \frac{12}{7}.$$

From $x_2 = 12/7 \Rightarrow 1 < x_2 < 2$.

$x_2 \leq 1$ or $x_2 \geq 2$.

Applying these condns seperately, we have to subproblems.

Sub problem 1 : Max $Z = 2x_1 + 2x_2$

Subject to $5x_1 + 3x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_2 \leq 1$ and $x_1, x_2 \geq 0$.

The optimal soln is Max $Z = 4$, $x_1 = 1$, $x_2 = 1$.

Sub problem 2 Max $Z = 2x_1 + 2x_2$

Subject to $5x_1 + 3x_2 \leq 8$

$x_1 + 2x_2 \le 4$

$x_2 \ge 2$ and $x_1, x_2 \ge 0$.

The optimal soln is Max $Z = 4$, $x_1 = 0$, $x_2 = 2$.

Original Problem.

Max $Z = 2x_1 + 2x_2$

Subject to

$5x_1 + 3x_2 \le 8$

$x_1 + 2x_2 \le 4$

and $x_1, x_2 \ge 0$.

Max $Z = 32/7$, $x_1 = 4/7$, $x_2 = \frac{12}{7}$

$x_2 \le 1$ $\qquad$ $x_2 \ge 2$.

Sub Pbm 1.

Max $Z = 4$

$x_1 = 1$, $x_2 = 1$

Fathomed

Sub Pbm 2

Max $Z = 4$

$x_1 = 0$, $x_2 = 2$

Fathomed,

4(a) Use Bellman's principle of optimality to maximize $b_1x_1 + b_2x_2 + \cdots + b_nx_n$ where $x_1 + x_2 + \cdots + x_n = c$,

$x_1, x_2 \ldots x_n \ge 0$

Soln: To develop the recursive equation.

Let $f_n(c)$ be the maximum attainable sum $b_1x_1 + \cdots + b_nx_n$ when the +ve constant c is divided into parts $x_1, x_2 \ldots x_n$.

For $n = 1$, let $x_1 = c$.

$\therefore f_1(c) = \underset{x_1 = c}{\text{Max}}\; b_1 x_1 = b_1 c.$

For $n=2$, Here c is divided into two parts $x_2 = z$, $x_1 = c-z$ such that $x_1 + x_2 = c$.

Then $$f_2(c) = \underset{0 \le x \le c}{\text{Max}} \{b_1 x_1 + b_2 x_2\}$$

$$= \underset{0 \le x \le c}{\text{Max}} \{b_1(c-z) + b_2 z\}.$$

$$= \underset{0 \le x \le c}{\text{Max}} \{b_2 z + f_1(c-z)\}.$$

For $n=3$, Here c is divided into three parts such that $x_1 + x_2 + x_3 = c$. $x_3 = z$ and $x_1 + x_2 = c-z$

(ie) $c-z$ divided into two parts whose maximum attainable sum is $f_2(c-z)$

$$f_3(c) = \underset{0 \le x \le c}{\text{Max}} \{b_1 x_1 + b_2 x_2 + b_3 x_3\}.$$

$$= \underset{0 \le x \le c}{\text{Max}} \{b_3 z + f_{21}(c-z)\}.$$

In general: the recursive eqn for the n-stage problem is

$$f_n(c) = \underset{0 \le x \le c}{\text{Max}} \{b_1 x_1 + b_2 x_2 + \dots + b_n x_n\}.$$

$$= \underset{0 \le x \le c}{\text{Max}} \{b_n z + f_{n-1}(c-z)\},$$

To solve the recursive eqn

For $n=2$.

$$f_2(c) = \max_{0 \le x \le c} \{ b_2 z + f_1(c-z) \}.$$

$$= \max_{0 \le x \le c} \{ b_2 z + b_1 (c-z) \}$$

$$= \max_{0 \le x \le c} \{ (b_2 - b_1) z + b_1 c \}$$

If $b_2 - b_1$ is +ve Then this is maximum for $z=c$, Otherwise it will be minimum

$$\therefore f_2(c) = \max_{0 \le x \le c} \{ (b_2 - b_1) z + b_1 c \} = b_2 c.$$

The optimum policy is $f_2(c) = b_2 c$.

For $n=3$

$$f_3(c) = \max_{0 \le x \le c} \{ b_3 z + f_2(c-z) \}.$$

$$= \max_{0 \le x \le c} \{ b_3 z + b_2 (c-z) \}.$$

$$= \max_{0 \le x \le c} \{ (b_3 - b_2) z + b_2 c \},$$

If $(b_3 - b_2)$ is positive, then this is maximum for $z=c$. Otherwise it will be minimum.

$$f_3(c) = \max_{0 \le x \le c} \{ (b_3 - b_2) z + b_2 c \}$$

$$= b_3 c.$$

$\therefore$ The optimal policy is $f_3(c) = b_3 c$.